

Sampling Policy Design for Tracking Time-varying Graph Signals with Adaptive Budget Allocation

Xuan Xie, Hui Feng, Bo Hu

Research Center of Smart Networks and Systems, School of Information Science and Technology

Fudan University, Shanghai, China

E-mail: {xxie15, hfeng, bohu}@fudan.edu.cn

Abstract—There have been many works that focus on the sampling policy design for static graph signals (GS), but few for time-varying GS. In this paper, we concentrate on how to select vertices to sample and how to allocate the sampling budget for a time-varying GS to reduce tracking error. In the Kalman Filter (KF) framework, the problem of sampling policy design and budget allocation is formulated as an infinite horizon sequential decision process, in which the optimal sampling policy is obtained by Dynamic Programming (DP). Since the optimal policy is intractable, an approximate algorithm is proposed by truncating the infinite horizon to two stages. By introducing a new tool for analyzing the convexity or concavity of composite functions, we prove that the truncated problem is convex so that it can be solved by standard tools. Finally, we demonstrate the performance of the proposed approach through numerical experiments.

Index Terms—Time-Varying Graph Signals, Sampling Policy Design, Kalman Filter, Dynamic Programming

I. INTRODUCTION

Time-varying graph signals (GS) are versatile for describing dynamics of signals in irregular domains, such as social, sensor and brain networks. There have been various works on stationariness, filtering and sampling of time-varying GS. For example, stationary processes of GS and some corresponding applications are investigated in [1], [2], frequency analysis and time-graph filter are proposed in [3]–[5], time-varying GS reconstruction and sampling are introduced in [6], [7].

For some large-scale networks, sampling theory for GS is essential since it is almost impractical to acquire the signals on all the nodes. Instead, the whole GS has to be estimated from the samples on a subset of nodes. For example, the opinions of all the users in a network is usually estimated by sending questionnaires to part of the users, since it is unaffordable to obtain the opinion of everyone in a huge social network due to the limited time and manpower.

When estimating GS from noisy observations, different sampling policies will result in different estimation performance. The design of sampling set aims at sampling vertices under the budget constraints to minimize the estimation error. There are many works that focus on the sampling set design for static GS [8]–[10]. Most existing sampling policy for time-varying GS minimize the tracking error myopically. In [11], sampling policies are designed for tracking bandlimited time-varying GS under least mean squares (LMS) and recursive least squares (RLS) framework. Sampling strategy for tracking bandlimited GS by Kalman Filter (KF) is proposed in [12]. In a related but

not identical scenario, sensor selection is designed for target tracking in the network by KF [13] and extended KF [14]. However, [11]–[14] do not consider the effect of the present sampling policy to the future tracking error, which may be not optimal for the long-term tracking performance.

The evolution of time-varying GS can depict both the change of signal like heat diffusion and the change of topology. For example, the diffusion of GS on a random edge sampling (RES) graph [15] which describes the case like link failures on the communication network, or street closures on the street network. Considering that the evolution may be slow or abrupt, *a good sampling policy design should not only focus on the instant tracking performance but also consider the long-term performance*. Meanwhile, a reasonable allocation of sampling budget among time steps will be also beneficial for tracking.

In KF framework, we consider the problem of sampling policy design for tracking a time-varying GS over an infinite horizon with a given average budget. Different from [11], [12], whose sampling sets are designed to minimize the instant tracking error, we consider the influence of current sampling policy to the future tracking performance and aim to minimize the tracking error for the long-term. Instead of given a fixed sampling budget for each time step, we also try to adaptively allocate the budget to minimize the tracking error. The problem of sampling set design and budget allocation is formulated as an infinite horizon sequential decision process and solved by dynamic programming (DP). Furthermore, an approximate optimization problem is proposed to get a suboptimal solution since the optimal solution is computationally prohibitive. We also prove that the approximate optimization problem is convex by introducing a new tool to analyze the convexity of composite matrix valued functions so that it can be solved by standard tools. Finally, several experiments validate that our approach has significantly improved the tracking performance than the state-of-art methods especially when the evolution of the GS is abrupt.

II. TIME-VARYING GRAPH SIGNALS

Consider an N -vertex undirected connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathbf{W})$, where \mathcal{V} is the vertex set, \mathcal{E} is the edge set, and \mathbf{W} is the weighted adjacency matrix. If there is an edge $e = (i, j)$ between vertex i and j , then $W_{i,j}$ represents the weight of the edge; otherwise $W_{i,j} = 0$. A time-varying graph signal

$\mathbf{f}_t \in \mathbb{R}^N$ at the moment t has the element $(\mathbf{f}_t)_i$ representing the signal value on the i -th vertex in \mathcal{V} .

The graph Laplacian is defined as $\mathbf{L} = \mathbf{D} - \mathbf{W}$, where the weighted degree matrix $\mathbf{D} = \text{diag}(\mathbf{1}^T \mathbf{W})$ and $\mathbf{1}$ is a vector with all ones. Since the Laplacian matrix is real symmetric, it has a complete eigenbasis and the spectral decomposition $\mathbf{L} = \mathbf{V} \mathbf{\Lambda} \mathbf{V}^T$, where the eigenvectors $\{\mathbf{u}_k\}_{0 \leq k \leq N-1}$ of \mathbf{L} form the columns of \mathbf{V} , and $\mathbf{\Lambda} \in \mathbb{C}^{N \times N}$ is a diagonal matrix of eigenvalues $0 = \lambda_0 \leq \lambda_1 \leq \dots \leq \lambda_{N-1}$ of \mathbf{L} . The Graph Fourier Transform (GFT) corresponds to the basis expansion of a signal. The eigenvectors of \mathbf{L} are regarded as the graph Fourier bases and the eigenvalues are regarded as frequencies [16]. The expansion coefficients of a static graph signal \mathbf{f} in terms of eigenvectors are defined as $\hat{\mathbf{f}}$, so that a GFT pair can be expressed as $\mathbf{f} = \mathbf{V} \hat{\mathbf{f}}$ and $\hat{\mathbf{f}} = \mathbf{V}^T \mathbf{f}$.

In this paper, we assume that the GS is a stochastic signal and the stochastic prior is usually given in frequency domain [2], [17], such that $\hat{\mathbf{f}}$ is drawn from the following distribution

$$p(\hat{\mathbf{f}}) \propto \exp(-(\hat{\mathbf{f}} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}_{\hat{\mathbf{f}}}^{-1} (\hat{\mathbf{f}} - \boldsymbol{\mu})), \quad (1)$$

where $p(\cdot)$ denotes probability density function, $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}_{\hat{\mathbf{f}}} = \text{diag}(\sigma_1^2, \dots, \sigma_N^2)$ are the mean and covariance matrix of $\hat{\mathbf{f}}$ respectively.

We assume that the time-varying GS follow a predefined evolution matrix \mathbf{H}_t , which can be used to depict the network dynamics in both GS and topology, for example, disease progression [18], opinion propagation [19] and topology with random edge connections [15]. The evolution noise is introduced to fit the uncertainty. Specifically, we have

$$\mathbf{f}_t = \mathbf{H}_t \mathbf{f}_{t-1} + \mathbf{v}_{t-1}, \quad \mathbf{v}_t \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_{\mathbf{v}}), \quad (2a)$$

$$\mathbf{y}_t = \boldsymbol{\Psi}_t (\mathbf{f}_t + \mathbf{w}_t), \quad \mathbf{w}_t \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_{\mathbf{w}}), \quad (2b)$$

where \mathbf{w}_t and \mathbf{v}_t are uncorrelated *i.i.d.* Gaussian noise, $\boldsymbol{\Sigma}_{\mathbf{v}} = \sigma_v^2 \mathbf{I}$ and $\boldsymbol{\Sigma}_{\mathbf{w}} = \sigma_w^2 \mathbf{I}$. Eq. (2a) represents the GS evolution model and (2b) is the observation model. The sampling operator $\boldsymbol{\Psi}_t : \mathbb{R}^N \mapsto \mathbb{R}^M$ is defined as

$$(\boldsymbol{\Psi}_t)_{i,j} = \begin{cases} 1, & \text{if } j\text{-th node is sampled at time } t; \\ 0, & \text{otherwise.} \end{cases} \quad (3)$$

By applying GFT, we change the GS to spectral domain and rewrite (2) as

$$\hat{\mathbf{f}}_t = \mathbf{V}^T \mathbf{H}_t \mathbf{V} \hat{\mathbf{f}}_{t-1} + \mathbf{V}^T \mathbf{v}_t = \tilde{\mathbf{H}}_t \hat{\mathbf{f}}_{t-1} + \mathbf{V}^T \mathbf{v}_t, \quad (4a)$$

$$\mathbf{y}_t = \boldsymbol{\Psi}_t (\mathbf{V} \hat{\mathbf{f}}_t + \mathbf{w}_t), \quad (4b)$$

where $\tilde{\mathbf{H}}_t = \mathbf{V}^T \mathbf{H}_t \mathbf{V}$.

The KF can be applied for tracking the GS described by (4), which consists of the following equations for each time step $t = 1, 2, \dots$:

$$\hat{\mathbf{f}}_t^- = \tilde{\mathbf{H}}_t \hat{\mathbf{f}}_{t-1}^+, \quad (5)$$

$$\mathbf{P}_t^- = \tilde{\mathbf{H}}_t \mathbf{P}_{t-1}^+ \tilde{\mathbf{H}}_t^T + \boldsymbol{\Sigma}_{\mathbf{v}}, \quad (6)$$

$$\mathbf{K}_t = \mathbf{P}_t^- \mathbf{V}^T \boldsymbol{\Psi}_t^T (\boldsymbol{\Psi}_t (\mathbf{V} \mathbf{P}_t^- \mathbf{V}^T + \boldsymbol{\Sigma}_{\mathbf{w}}) \boldsymbol{\Psi}_t^T)^{-1}, \quad (7)$$

$$\hat{\mathbf{f}}_t^+ = \hat{\mathbf{f}}_t^- + \mathbf{K}_t (\mathbf{y}_t - \boldsymbol{\Psi}_t \mathbf{V} \hat{\mathbf{f}}_t^-), \quad (8)$$

$$\mathbf{P}_t^+ = ((\mathbf{P}_t^-)^{-1} + \mathbf{V}^T \boldsymbol{\Psi}_t^T \boldsymbol{\Psi}_t \boldsymbol{\Sigma}_{\mathbf{w}}^{-1} \boldsymbol{\Psi}_t^T \mathbf{V})^{-1}, \quad (9)$$

where $\hat{\mathbf{f}}_t^-$, \mathbf{P}_t^- , \mathbf{K}_t , $\hat{\mathbf{f}}_t^+$, \mathbf{P}_t^+ denote the prior estimation, prior covariance, KF gain, posterior estimation and posterior covariance respectively. The initialization states are $\hat{\mathbf{f}}_0^+ = \boldsymbol{\mu}$ and $\mathbf{P}_0^+ = \boldsymbol{\Sigma}_{\hat{\mathbf{f}}}$ based on (1).

III. SAMPLING POLICY DESIGN

A. Sampling as an Infinite Horizon Decision Process

In the KF framework, the mean squared error (MSE) of GS estimation in time step t can be calculated by $\text{tr}(\mathbf{P}_t^+)$. In order to get an optimal tracking performance for an infinite horizon, we design a sequence of sampling operators $\{\boldsymbol{\Psi}_1, \boldsymbol{\Psi}_2, \dots\}$ that minimize the accumulated tracking error $\sum_{t=1}^{\infty} \gamma^t \text{tr}(\mathbf{P}_t^+)$ under the sampling budget constraints, where $\gamma \in (0, 1)$ is the discount factor. The effect of $\boldsymbol{\Psi}_t$ to the accumulated tracking error is cascading according to (9) and (6), so the sampling operator design at present time must balance the present tracking error and the future tracking error.

Denote $(\mathbf{P}_t^-)^{-1}$ and \mathbf{D}_t as the state and action of the system at time step t respectively, where $\mathbf{D}_t \triangleq \boldsymbol{\Psi}_t^T \boldsymbol{\Psi}_t = \text{diag}(d_{1,t}, \dots, d_{N,t})$ is a diagonal matrix with the i -th diagonal element equalling 1 if the i -th vertex is sampled, and 0 elsewhere. The decision process can be formulated as a 5-tuple $(\mathcal{S}, \mathcal{A}, f_t, g_t, \gamma)$, where \mathcal{S} is the state set of symmetric positive definite matrices $(\mathbf{P}_t^-)^{-1}$, \mathcal{A} is the action set of $N \times N$ matrices \mathbf{D}_t , f_t is the law of the state transition with the form of $(\mathbf{P}_{t+1}^-)^{-1} = f_t((\mathbf{P}_t^-)^{-1}, \mathbf{D}_t)$. According to (9) and (6) in KF, the state transition guided by \mathbf{D}_t is

$$(\mathbf{P}_{t+1}^-)^{-1} = [\tilde{\mathbf{H}}_{t+1} ((\mathbf{P}_t^-)^{-1} + \sigma_w^{-2} \mathbf{V}^T \mathbf{D}_t \mathbf{V})^{-1} \tilde{\mathbf{H}}_{t+1}^T + \boldsymbol{\Sigma}_{\mathbf{v}}]^{-1}.$$

g_t is the immediate cost of action defined as the estimation error of instant estimation with the form of $g_t((\mathbf{P}_t^-)^{-1}, \mathbf{D}_t)$, which is affected by the present sampling policy \mathbf{D}_t ,

$$g_t((\mathbf{P}_t^-)^{-1}, \mathbf{D}_t) = \text{tr}(\mathbf{P}_t^+) = \text{tr}((\mathbf{P}_t^-)^{-1} + \sigma_w^{-2} \mathbf{V}^T \mathbf{D}_t \mathbf{V})^{-1}.$$

Then the optimal problem for sampling policy design can be formulated as choosing a sequence of actions in order to minimize the total cost over an infinite horizon,

$$\begin{aligned} \min_{\mathbf{D}_1, \mathbf{D}_2, \dots} \quad & J = \sum_{t=1}^{\infty} \gamma^t g_t((\mathbf{P}_t^-)^{-1}, \mathbf{D}_t) \\ \text{s.t.} \quad & 0 \leq \text{tr}(\mathbf{D}_t) \leq M_t \quad t = 1, 2, \dots, \\ & \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T \text{tr}(\mathbf{D}_t) = M, \\ & \mathbf{D}_t \subset \mathcal{A} \quad t = 1, 2, \dots \end{aligned} \quad (10)$$

The Bellman equation [20] is used to compute the optimal action for the decision process sequentially,

$$\begin{aligned} J_t((\mathbf{P}_t^-)^{-1}) = \min_{\mathbf{D}_t} \{ & g_t((\mathbf{P}_t^-)^{-1}, \mathbf{D}_t) \\ & + \gamma J_{t+1}(f_{t+1}((\mathbf{P}_t^-)^{-1}, \mathbf{D}_t)) \}, \end{aligned} \quad (11)$$

which means the optimal policy at t is the one that minimizes the sum of immediate cost and future costs.

B. The Truncated Problem

However, finding an optimal solution for (11) is computational intractable. One reason is that the dimension of action space grows exponentially with the tracking time t . According to (10), the weight of immediate cost in the total cost decreases over time, which means the cost in the near future has a bigger impact on the total cost. So we truncate the infinite horizon future cost in (11) to the length of one [21], [22], which means the policy of each time step only minimizes the sum of immediate cost and the cost of the next time. For each time step t , the future cost is truncated to the optimal cost of $t + 1$ as

$$\begin{aligned} J_{t+1}((\mathbf{P}_{t+1}^-)^{-1}) &= \min_{\mathbf{D}_{t+1}} g_{t+1}((\mathbf{P}_{t+1}^-)^{-1}, \mathbf{D}_{t+1}) \\ &= \min_{\mathbf{D}_{t+1}} \text{tr}((\mathbf{P}_{t+1}^-)^{-1} + \sigma_{\mathbf{w}}^{-2} \mathbf{V}^T \mathbf{D}_{t+1} \mathbf{V})^{-1}. \end{aligned}$$

Thus, we obtain a new one-step-look-ahead object function,

$$\begin{aligned} J_t((\mathbf{P}_t^-)^{-1}) &= \min_{\mathbf{D}_t} \left\{ \text{tr}((\mathbf{P}_t^-)^{-1} + \sigma_{\mathbf{w}}^{-2} \mathbf{V}^T \mathbf{D}_t \mathbf{V})^{-1} \right. \\ &\quad \left. + \gamma \min_{\mathbf{D}_{t+1}} \left\{ \text{tr}(f_t((\mathbf{P}_t^-)^{-1}, \mathbf{D}_t)) + \sigma_{\mathbf{w}}^{-2} \mathbf{V}^T \mathbf{D}_{t+1} \mathbf{V} \right\}^{-1} \right\}. \end{aligned} \quad (12)$$

Also truncating allocation of the sampling budget to two time steps, the new optimization problem is as follow

$$\begin{aligned} \min_{\tilde{\mathbf{D}}_t, \tilde{\mathbf{D}}_{t+1}} \quad & \text{tr}((\mathbf{P}_t^-)^{-1} + \sigma_{\mathbf{w}}^{-2} \mathbf{V}^T \tilde{\mathbf{D}}_t \mathbf{V})^{-1} + \\ & \gamma \text{tr}(f_t((\mathbf{P}_t^-)^{-1}, \tilde{\mathbf{D}}_t) + \sigma_{\mathbf{w}}^{-2} \mathbf{V}^T \tilde{\mathbf{D}}_{t+1} \mathbf{V})^{-1} \quad (13) \\ \text{s.t.} \quad & 0 \leq \text{tr}(\tilde{\mathbf{D}}_t) \leq M_t, \\ & 0 \leq \text{tr}(\tilde{\mathbf{D}}_{t+1}) \leq M_{t+1}, \\ & \text{tr}(\tilde{\mathbf{D}}_t) + \text{tr}(\tilde{\mathbf{D}}_{t+1}) = 2M, \\ & \tilde{\mathbf{D}}_t, \tilde{\mathbf{D}}_{t+1} \subset \tilde{\mathcal{A}}, \end{aligned}$$

where $\tilde{\mathcal{A}}$ is a set of $N \times N$ matrices $\tilde{\mathbf{D}}_t$ with element $0 \leq d \leq 1$ in the diagonal line and 0 everywhere else. Compared with problem (10), we relax $d_{i,t}$ and $d_{i,t+1}$ to continuous values in $[0, 1]$ since the optimization problem is an intractable combinatorial problem before relaxing. The design of sampling policy for KF is described in Algorithm 1.

Algorithm 1 Sampling policy design and GS tracking.

- 1: **Initialize** $\hat{\mathbf{f}}_0^+$ and \mathbf{P}_0^+ .
 - 2: **for** $t = 1, 3, 5, \dots$ **do**
 - 3: Update $\hat{\mathbf{f}}_t^+$, \mathbf{P}_t^+ by (5) and (6);
 - 4: Solve optimization problem (13) to get $\tilde{\mathbf{D}}_t$ and $\tilde{\mathbf{D}}_{t+1}$;
 - 5: Calculate the sampling budget of time step t by $M_t^* = \text{round}(\text{tr}(\tilde{\mathbf{D}}_t))$ and the sampling budget of time step $t+1$ by $M_{t+1}^* = 2M - M_t^*$;
 - 6: Sampling the M_t^* and M_{t+1}^* vertices with largest $d_{i,t}$ and $d_{i,t+1}$ in time step t and $t + 1$, respectively;
 - 7: Update $\hat{\mathbf{f}}_t^+$ and \mathbf{P}_t^+ by (8) and (9).
 - 8: **end for**
-

IV. ANALYSIS

The optimal relaxed solution for $(\tilde{\mathbf{D}}_t, \tilde{\mathbf{D}}_{t+1})$ in (13) can be solved by any standard optimization tool if it is convex. So in this section, we are going to analyze the convexity of object function (13).

A. Convexity Composition for Matrix Valued Functions

Object function (13) is a composite function of $(\tilde{\mathbf{D}}_t, \tilde{\mathbf{D}}_{t+1})$. Usually, the convexity of composition function is analyzed by the second derivative as in [23, Sec. 3.2.4]. But it is hard to calculate the derivative for the matrix valued function, so we propose a new tool to analyze the convexity.

For a symmetric matrix \mathbf{X} , $\mathbf{X} \succeq (\preceq) \mathbf{0}$ means it is positive (negative) semidefinite. For two positive or negative semidefinite matrices $\mathbf{X}_1, \mathbf{X}_2$, $\mathbf{X}_1 \succeq (\preceq) \mathbf{X}_2$ means matrix $\mathbf{X}_1 - \mathbf{X}_2$ is positive or negative semidefinite. Suppose $f : \mathbf{S}_{+(-)}^n \mapsto \mathbf{S}_{+(-)}^m$ is a matrix valued function, where $\mathbf{S}_{+(-)}^n$ denotes the set of symmetric positive (negative) semidefinite $n \times n$ matrices.

Definition 1: Function f is matrix nonincreasing (non-decreasing) if $f(\mathbf{X}_1) \preceq (\succeq) f(\mathbf{X}_2)$ for $\mathbf{X}_1 \succeq \mathbf{X}_2$.

Definition 2: [23, Sec. 3.6.2] Function f is matrix convex (concave) with respect to matrix inequality if

$$f(\theta \mathbf{X}_1 + (1 - \theta) \mathbf{X}_2) \preceq (\succeq) \theta f(\mathbf{X}_1) + (1 - \theta) f(\mathbf{X}_2)$$

for $\mathbf{X}_1, \mathbf{X}_2 \in \mathbf{S}_+^n$ or $\mathbf{X}_1, \mathbf{X}_2 \in \mathbf{S}_-^n$ and any $\theta \in [0, 1]$.

Theorem 1 (Rule 1): Let $h_s : \mathbf{S}_{+(-)}^m \mapsto \mathbf{S}_{+(-)}^k$ and $h_m : \mathbf{S}_{+(-)}^n \mapsto \mathbf{S}_{+(-)}^m$, $h = h_s \circ h_m$ is matrix convex if h_s is matrix convex and nonincreasing and h_m is matrix concave.

Proof: We can obtain the following inequalities

$$\begin{aligned} & h_s(h_m(\theta \mathbf{X}_1 + (1 - \theta) \mathbf{X}_2)) \\ & \preceq h_s(\theta h_m(\mathbf{X}_1) + (1 - \theta) h_m(\mathbf{X}_2)) \\ & \preceq \theta h_s(h_m(\mathbf{X}_1)) + (1 - \theta) h_s(h_m(\mathbf{X}_2)), \end{aligned} \quad (14)$$

where the first inequality comes from the matrix concavity of h_m and matrix nonincreasing property of h_s , and the second inequality comes from the matrix convexity of h_s . Thus, the theorem is proved. ■

When $k = 1$, h_s will be a scalar valued function and the ' \preceq ' in the second line of (14) will become ' \leq '. We can also get the other three composition rules using the similar method as follow:

- Rule 2: $h = h_s \circ h_m$ is matrix convex if h_s is matrix convex and nondecreasing and h_m is matrix convex.
- Rule 3: $h = h_s \circ h_m$ is matrix concave if h_s is matrix concave and nonincreasing and h_m is matrix convex.
- Rule 4: $h = h_s \circ h_m$ is matrix concave if h_s is matrix concave and nondecreasing and h_m is matrix concave.

These composition rules can be applied multiple times to analyze the convexity or concavity of matrix valued functions.

B. Convexity of Object Function

By using Theorem 1, we can get the following lemmas.

Lemma 1: $h_1(\mathbf{X}) = \text{tr}(\mathbf{X}^{-1})$ is convex and nonincreasing for $\mathbf{X} \in \mathbf{S}_{++}^n$. $\tilde{h}_1(\mathbf{X}) = -\text{tr}(\mathbf{X}^{-1})$ is also convex and

nonincreasing for $\mathbf{X} \in \mathbf{S}_{--}^n$, where $\mathbf{S}_{++(-)}^n$ denotes a set of symmetric positive (negative) definite matrices.

Lemma 2: $h_2(\mathbf{X}) = -\mathbf{A}^T \mathbf{X}^{-1} \mathbf{A} - \mathbf{B}$ is concave and nondecreasing for $\mathbf{X}, \mathbf{A}, \mathbf{B} \in \mathbf{S}_{++}^n$. $\tilde{h}_2(\mathbf{X}) = \mathbf{A}^T \mathbf{X}^{-1} \mathbf{A} + \mathbf{B}$ is also concave and nondecreasing for $\mathbf{X}, \mathbf{A}, \mathbf{B} \in \mathbf{S}_{--}^n$.

Theorem 2: The object function in (13) is a convex function of the relaxed $(\tilde{\mathbf{D}}_t, \tilde{\mathbf{D}}_{t+1})$.

Proof: See Appendix A. ■

V. NUMERICAL RESULTS

We now numerically evaluate the performance of the proposed work. The experiments compare the proposed work with the following three methods: M1 [13], M2 [12] and random sampling by normalized MSE (NMSE)

$$\text{NMSE}(t) = \frac{\|\hat{\mathbf{f}}_t^+ - \hat{\mathbf{f}}_t\|_2^2}{\|\hat{\mathbf{f}}_t\|_2^2}.$$

M1 only considers the generalized information gain which is only related to observation model but ignores the signal evolution. M2 takes signal evolution into consideration but ignores the long-term performance.

We first simulate the process of a heat source moving in a sensor network. The sensor network is modelled as a graph with 100 vertices randomly placed in a unit square and the edges exist between vertices of which the distance is no more than 0.6. The heat source moves in a given trajectory which is generated by a random walk. The evolution matrix of GS is given by the a graph translation [24] operator according to the trajectory. For example, if the center vertices of the trajectory for two continuous time steps are vertex a and b , the evolution matrices will be $\tilde{H}_t = \text{diag}(\mathbf{V}^T \delta_a)$ and $\tilde{H}_{t+1} = \text{diag}(\mathbf{V}^T \delta_b)$. The energy of the GS at each time step is normalized to 1. The evolution and observation noise are *i.i.d* zero-mean Gaussian white noise with $\sigma_v^2 = 10^{-4}$ and $\sigma_w^2 = 10^{-3}$. The initialization states of the GS are $\hat{\mathbf{f}}_0^+ = \mathbf{1}_{N \times 1}$ and $\mathbf{P}_0^+ = \mathbf{I}_{N \times N}$. The discount factor γ is set to $\gamma = 0.8$. The average sampling budget $M = 10$ and the largest budget of each time $M_t = 20$. For the compared algorithms, the sampling budget of each time step is fixed to 10. The accumulated tracking error for 1000 time steps is shown in the second line of Table. I. The step-by-step tracking performance of the first 100 time steps is shown in Fig. 1.

We can find that M1 and random sampling almost lose tracking of the GS and the proposed algorithm improves the tracking performance significantly compared to M2 when the signal evolution between two time steps is abrupt. A visualized demonstration of the abrupt heat source translation from time step 31 to 32 is shown in Fig. 2 (a) (b), the circled vertices are the sampled vertices among which the center vertex is in the full line circle and the others are in dashed line circles. It can be seen that since the long-term performance is considered, the proposed algorithm allocates more sampling budget to time step 32 to make the estimation at step 32 more accuracy.

Next, we show that our sampling policy also fits for the tracking of GS on time-varying topology, like RES graph. The opinion evolution in the social network is taken as an example.

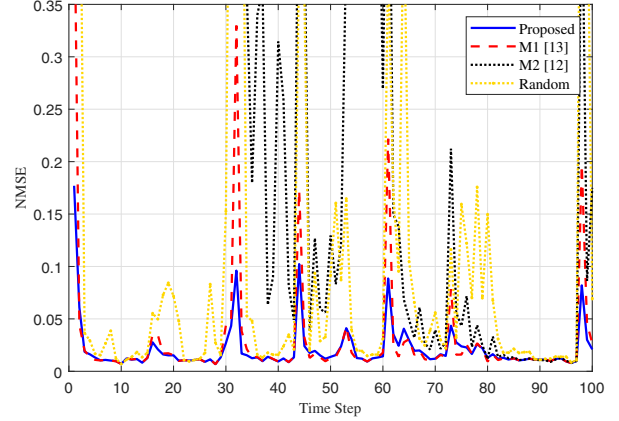


Fig. 1. Tracking performance of different algorithms on sensor network.

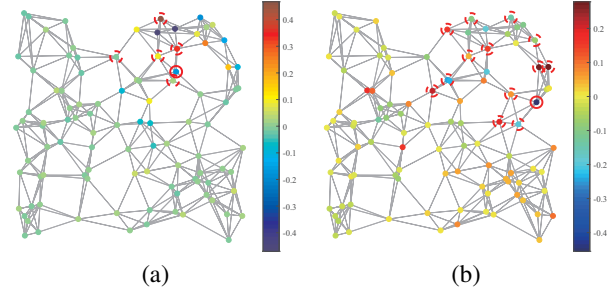


Fig. 2. GS and sampling sets on sensor network at time step 31 (a) and 32 (b).

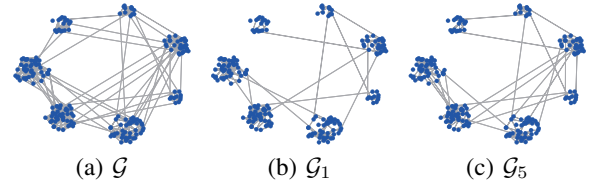


Fig. 3. Example of RES community graphs.

A community graph with 7 communities is used to model a social network. The probability that a edge $e = (i, j)$ in the edge set \mathcal{E} is activated at time t is set to $p_{i,j} = 0.5$. The edges are activated independently across time. At each time step t , we draw a graph realization $\mathcal{G}_t = (\mathcal{V}, \mathcal{E}_t)$ from the underlying graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where the edge set $\mathcal{E}_t \subseteq \mathcal{E}$ is generated via an *i.i.d*. Bernoulli process. An example of the RES community graph is shown in Fig. 3.

The opinion dynamics of individuals in the network follows the Krause-Hegselmann's model [25], which consider the opinion evolution of the individuals as a weighted average of their opinions at a previous time with bounded confidence.

TABLE I
ACCUMULATED TRACKING ERROR OF DIFFERENT ALGORITHMS.

	Proposed	M1 [13]	M2 [12]	Random
Sensor network (1000 time steps)	22.412	460.124	26.582	151.973
Social network (100 time steps)	0.7186	0.8004	0.7385	0.7865

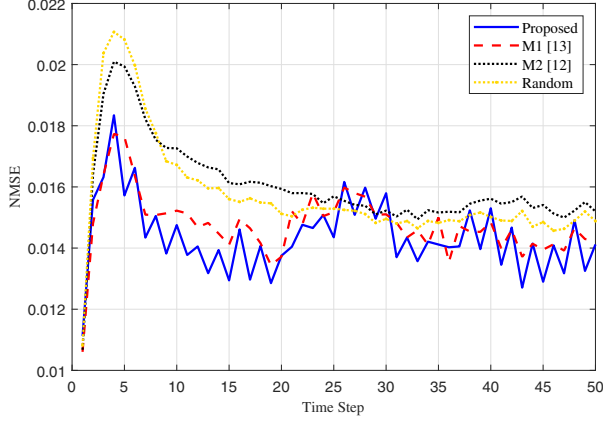


Fig. 4. Tracking performance of different algorithms on social network.

The evolution of the GS on vertex i at time step t follows

$$(\mathbf{f}_t)_i = |I(i, \mathbf{f}_{t-1})|^{-1} \sum_{j \in I(i, \mathbf{f}_{t-1})} (\mathbf{f}_{t-1})_j, \quad (15)$$

$$\begin{aligned} & \gamma \text{tr} \left(\left(\tilde{\mathbf{H}}_t \left((\mathbf{P}_t^-)^{-1} + \sigma_{\mathbf{w}}^{-2} \mathbf{V}^T \tilde{\mathbf{D}}_t \mathbf{V} \right)^{-1} \tilde{\mathbf{H}}_t + \Sigma_{\mathbf{v}} \right)^{-1} + \sigma_{\mathbf{w}}^{-2} \mathbf{V}^T \tilde{\mathbf{D}}_{t+1} \mathbf{V} \right)^{-1} \\ &= -\gamma \text{tr} \left(\left(-\tilde{\mathbf{H}}_t \left((\mathbf{P}_t^-)^{-1} + \sigma_{\mathbf{w}}^{-2} \mathbf{V}^T \tilde{\mathbf{D}}_t \mathbf{V} \right)^{-1} \tilde{\mathbf{H}}_t - \Sigma_{\mathbf{v}} \right)^{-1} - \sigma_{\mathbf{w}}^{-2} \mathbf{V}^T \tilde{\mathbf{D}}_{t+1} \mathbf{V} \right)^{-1}. \end{aligned} \quad (16)$$

$$\begin{aligned} & \left(-\tilde{\mathbf{H}}_t \left((\mathbf{P}_t^-)^{-1} + \sigma_{\mathbf{w}}^{-2} \mathbf{V}^T (\theta \mathbf{X}_1 + (1-\theta) \mathbf{Y}_1) \mathbf{V} \right)^{-1} \tilde{\mathbf{H}}_t - \Sigma_{\mathbf{v}} \right)^{-1} - \sigma_{\mathbf{w}}^{-2} \mathbf{V}^T (\theta \mathbf{X}_2 + (1-\theta) \mathbf{Y}_2) \mathbf{V} \\ & \geq \theta \left(-\tilde{\mathbf{H}}_t \left((\mathbf{P}_t^-)^{-1} + \sigma_{\mathbf{w}}^{-2} \mathbf{V}^T \mathbf{X}_1 \mathbf{V} \right)^{-1} \tilde{\mathbf{H}}_t - \Sigma_{\mathbf{v}} \right)^{-1} + (1-\theta) \left(-\tilde{\mathbf{H}}_t \left((\mathbf{P}_t^-)^{-1} + \sigma_{\mathbf{w}}^{-2} \mathbf{V}^T \mathbf{Y}_1 \mathbf{V} \right)^{-1} \tilde{\mathbf{H}}_t - \Sigma_{\mathbf{v}} \right)^{-1} \\ & \quad - \theta \sigma_{\mathbf{w}}^{-2} \mathbf{V}^T \mathbf{X}_2 \mathbf{V} - (1-\theta) \sigma_{\mathbf{w}}^{-2} \mathbf{V}^T \mathbf{Y}_2 \mathbf{V}. \end{aligned} \quad (17)$$

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APPENDIX A THE PROOF OF THEOREM 2

Theorem 2: The object function in (13) is a convex function of the relaxed $(\tilde{\mathbf{D}}_t, \tilde{\mathbf{D}}_{t+1})$.

where $I(i, \mathbf{f}_t) = \{1 \leq j \leq N \mid |(\mathbf{f}_t)_i - (\mathbf{f}_t)_j| \leq \epsilon\}$ and $|I(i, \mathbf{f}_t)|$ denotes the cardinality of $I(i, \mathbf{f}_t)$. The opinions of individuals are initialized by uniform random numbers in $[0, 1]$ with $\mathbf{P}_0^+ = 0.1 \mathbf{I}_{N \times N}$ and ϵ is set to 0.3. The energy of the GS at each time step is also normalized to 1. The evolution and observation noise are *i.i.d.* zero-mean Gaussian white noise with $\sigma_{\mathbf{v}}^2 = 10^{-4}$ and $\sigma_{\mathbf{w}}^2 = 10^{-4}$. The average sampling budget is set to $M = 10$ and the largest budget of each time step is $M_t = 20$, and the discount factor is set to $\gamma = 0.8$. The accumulated tracking error for 100 time steps is shown in the third line of Table. I, and the step-by-step tracking performance is shown in Fig. 4. In this case, the opinions on the social network become more and more smooth with time passing by according to (15), and therefore a more accurate estimation of GS in the former time step will help to estimate the GS in the later time step more accurately. By allocating more samples to the former time step, algorithm 1 also performance better in the long-term compared to the other methods with fixed sampling budget in each time step.

VI. CONCLUSION

In this paper, a sampling policy with adaptive budget allocation is proposed for tracking a time-varying graph signal with KF. By considering the influence of the current sampling policy to the future performance, we formulate the problem as an infinite horizon sequential decision process. An approximate solution is obtained by truncating the future horizon to one, which improves the tracking performance a lot.

Proof: Obviously, $\sigma_{\mathbf{w}}^{-2} \mathbf{V}^T \tilde{\mathbf{D}}_t \mathbf{V}$ is a linear function of $\tilde{\mathbf{D}}_t$. Since $(\mathbf{P}_t^-)^{-1} + \sigma_{\mathbf{w}}^{-2} \mathbf{V}^T \tilde{\mathbf{D}}_t \mathbf{V} \in \mathbf{S}_{+}^N$, using Lemma 1 and composition Rule 1 we can prove that the first term of (13) is a convex function of $(\tilde{\mathbf{D}}_t, \tilde{\mathbf{D}}_{t+1})$.

The second term of (13) can be rewritten as (16). For easier reading, let

$$\mathbf{Z}_1 = -\tilde{\mathbf{H}}_t \left((\mathbf{P}_t^-)^{-1} + \sigma_{\mathbf{w}}^{-2} \mathbf{V}^T \tilde{\mathbf{D}}_t \mathbf{V} \right)^{-1} \tilde{\mathbf{H}}_t - \Sigma_{\mathbf{v}}. \quad (18)$$

Using Lemma 2 and composition Rule 4, we can prove that \mathbf{Z}_1 is a concave function of $\tilde{\mathbf{D}}_t$.

It is obvious that \mathbf{Z}_1 is symmetric negative semidefinite. So we can prove that \mathbf{Z}_1^{-1} is a concave function of $\tilde{\mathbf{D}}_t$ using

Lemma 2 and composition Rule 4.

Since $\sigma_w^{-2} \mathbf{V}^T \tilde{\mathbf{D}}_{t+1} \mathbf{V}$ is a linear function of $\tilde{\mathbf{D}}_{t+1}$, for $\mathbf{X}_1, \mathbf{Y}_1, \mathbf{X}_2, \mathbf{Y}_2 \subset \tilde{\mathcal{A}}$ and $\theta \in [0, 1]$, we have the (17). According to (18), let

$$\mathbf{Z}_2 = \mathbf{Z}_1^{-1} - \sigma_w^{-2} \mathbf{V}^T \tilde{\mathbf{D}}_{t+1} \mathbf{V}. \quad (19)$$

Thus, \mathbf{Z}_2 is a concave function of $(\tilde{\mathbf{D}}_t, \tilde{\mathbf{D}}_{t+1})$.

Since $\mathbf{Z}_2 \in \mathbf{S}_{-}^N$, we can prove that the second term of (13) is a concave function of $(\tilde{\mathbf{D}}_t, \tilde{\mathbf{D}}_{t+1})$ using Lemma 1 and composition Rule 4. ■

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